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# BALANCE EQUATIONS FOR FLUID LINES, SHEETS, FILAMENTS AND MEMBRANES

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**Abstract**—Fluid structures can be classified into two categories: fluid filaments and fluid membranes. If the cross section of a fluid filament vanishes, the fluid filament is called a fluid line, whereas if the thickness of a fluid membrane vanishes, the fluid membrane is called a fluid sheet. In this paper, the local instantaneous balance equations are derived for a line and a sheet of fluid moving in a three-dimensional geometrical space. For a filament, the balance equations are obtained by using quantities averaged over the cross section of the filament. For a membrane, the balance equations are obtained by using quantities averaged over the height of the membrane. It is shown that the local and averaged formulations are consistent. All these balance equations have been rigorously derived by applying original mathematical theorems given in the appendices to the paper. Such balance equations can be used to model single-phase flow in bends or coils, the dynamic centering of thin liquid shells in capillary oscillations and the instability of an annular jet.

*Key Words:* liquid filaments, liquid membranes, averaged equations, liquid shells, annular jets

## 1. INTRODUCTION

The instability of a thin liquid jet can be studied by using the mass and momentum balance equations averaged over the cross section of the jet (Meier *et al.* 1992). The dynamics of a flame front has been modeled by considering the flame front as a surface moving in a three-dimensional space (Candel & Poinsot 1990). These two examples among others show that the dynamics of fluid structures such as filaments or membranes is of interest in many different fields of engineering.

Fluid structures can be classified into two categories: fluid *filaments* and fluid *membranes*. If the cross section of a fluid filament vanishes, the fluid filament will be called a fluid *line*, whereas if the thickness of a fluid membrane vanishes, the fluid membrane will be called a fluid *sheet*.

The derivation of the balance equations for the above-defined fluid structures is not a trivial problem. As will be seen in the following, several authors have tried to derive some of these balance equations but their derivations lead to incomplete equations (Zak 1979) or deal only with the mass balance for a sheet (Candel & Poinsot 1990).

It is the purpose of this paper to establish the mass and momentum balance equations rigorously. In section 2, one- and two-dimensional fluid structures, namely lines and sheets, are considered. Section 3 deals with the corresponding three-dimensional structures: filaments and membranes. In section 4, two types of modeling are proposed for thin structures such as thin filaments and thin membranes. The first one is a one-dimensional model based on a one-dimensional approximation. The balance equations are obtained from the equations of section 2, where the line (or surface) density is replaced by the product of a constant volumetric density by a cross-section area (or a height). In the second model, the equations are obtained by asymptotically reducing the thickness of the filament or the membrane in the equations derived in section 3. We will then show that the two models are completely equivalent. Finally, section 5 deals with some applications of the balance equations established in sections 2–4: one-dimensional modeling of single-phase flow in bends or coils, dynamic centering of thin liquid shells in capillary oscillations and instability of an annular jet.

## 2. THE FLUID LINE AND THE FLUID SHEET

### *2.1. Definitions*

*2.1.1. The fluid line*

A *fluid line* (figure 1) is a moving material line represented in space by a curve $(\Gamma_t)$. Every material point M in the fluid line is determined by

$$\mathbf{OM} = \mathbf{r}[p(t), t], \tag{1}$$

where $p$ is a parameter. The density of the line (mass per unit length) will be denoted by $\rho_l$ and the unit tangent vector of the curve $(\Gamma_t)$ by $\boldsymbol{\tau}$.

*2.1.2. The fluid sheet*

A *fluid sheet* (figure 2) is a moving material surface represented in space by a surface $(S_t)$. A curvilinear coordinate system $(x^1, x^2)$ is chosen on the surface. Every material point on the fluid sheet is determined by

$$\mathbf{OM} = \mathbf{r}[x^1(t), x^2(t), t]. \tag{2}$$

The density of the sheet (mass per unit area) will be denoted by $\rho_s$.

### *2.2. Mass Balance*

*2.2.1. The fluid line*

The mass balance for a *material* line of finite length AB reads (figure 3):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{AB}} \rho_l \,\mathrm{d}s = 0, \tag{3}$$

where $\mathrm{d}s$ is the elementary arc length.

Equation [3] can be transformed by using an original transport theorem for line integrals ([A3] in appendix A) into the following form:

$$\int_{\mathrm{AB}} \left[\frac{\partial \rho_l}{\partial t} + \mathbf{V}\cdot\boldsymbol{\tau}\frac{\partial \rho_l}{\partial s} + \rho_l \frac{\partial \mathbf{V}}{\partial s}\cdot\boldsymbol{\tau}\right]\mathrm{d}s = 0, \tag{4}$$

where $\mathbf{V}$ denotes the velocity of a material point of the fluid line. Therefore, it can be deduced that

$$\frac{\partial \rho_l}{\partial t} + \boldsymbol{\tau}\cdot\frac{\partial}{\partial s}(\rho_l \mathbf{V}) = 0. \tag{5}$$

In order to compare our equations with similar equations proposed previously, another form of this relation can be given by introducing the velocity of a geometrical point attached to the line $\mathbf{U}_\Gamma$:

$$\mathbf{U}_\Gamma \triangleq \left(\frac{\partial \mathbf{r}}{\partial t}\right) \quad \text{with fixed } p. \tag{6}$$

Equation [5] now reads:

$$\frac{\partial \rho_l}{\partial t} + \frac{\partial \rho_l}{\partial s}\boldsymbol{\tau}\cdot\mathbf{U}_\Gamma + \rho_l\boldsymbol{\tau}\cdot\frac{\partial \mathbf{U}_\Gamma}{\partial s} + \frac{\partial}{\partial s}[\rho_l(\mathbf{V} - \mathbf{U}_\Gamma)\cdot\boldsymbol{\tau}] = 0. \tag{7}$$

*2.2.2. The fluid sheet*

We consider a *material* surface $(\mathscr{S})$ (figure 4), defined as a part of the surface $(S_t)$ limited by a curve $(C)$. The mass balance reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{S}} \rho_s \,\mathrm{d}a = 0. \tag{8}$$

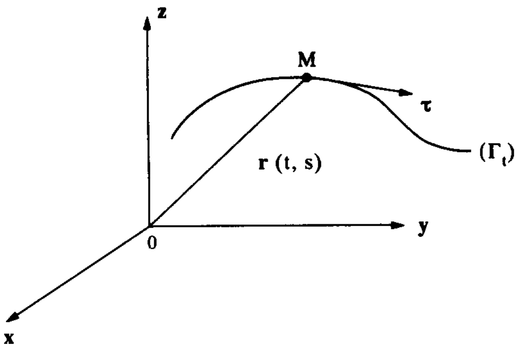


Figure 1. The fluid line.

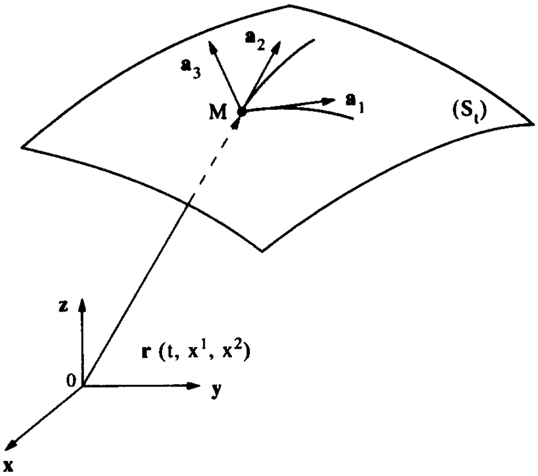


Figure 2. The fluid sheet.

Equation [8] is transformed by means of the Reynolds transport theorem for a material surface (Aris 1962) into the following form:

$$\int_{\mathscr{S}} \left( \frac{\partial \rho_s}{\partial t} + \mathbf{V} \cdot \mathbf{grad}_s \, \rho_s + \rho_s \, \mathrm{div}_s \, \mathbf{V} \right) \mathrm{d}a = 0, \tag{9}$$

where the vector $\mathbf{V}$ denotes the velocity of a material point of the fluid sheet, $\mathrm{div}_s$ denotes the surface divergence and $\mathbf{grad}_s$ the surface gradient defined in appendix B.

It can then be deduced that

$$\frac{\partial \rho_s}{\partial t} + \mathbf{V} \cdot \mathbf{grad}_s \, \rho_s + \rho_s \, \mathrm{div}_s \, \mathbf{V} = 0. \tag{10}$$

As the velocity $\mathbf{U}_s$ of a point attached to the fluid sheet is defined by

$$\mathbf{U}_s \triangleq \left( \frac{\partial \mathbf{r}}{\partial t} \right) \quad \text{with } x^1 \text{ and } x^2 \text{ fixed,} \tag{11}$$

equation [10] now reads:

$$\frac{\partial \rho_s}{\partial t} + \mathbf{U}_s \cdot \mathbf{grad}_s \, \rho_s + \rho_s \, \mathrm{div}_s \mathbf{U}_s + \mathrm{div}_s \, \rho_s (\mathbf{V} - \mathbf{U}_s)^* = 0, \tag{12}$$

where $(\mathbf{V} - \mathbf{U}_s)^*$ is the projection of the vector $(\mathbf{V} - \mathbf{U}_s)$ on the tangential plane.

*Remark.* Equations [5] and [10] can be put into the same form taking into account the definition of the operator $\mathrm{div}_l$ (appendix B):

$$\frac{\partial \rho_l}{\partial t} + \mathrm{div}_l(\rho_l \mathbf{V}) = 0 \tag{13}$$

and

$$\frac{\partial \rho_s}{\partial t} + \mathrm{div}_s(\rho_s \mathbf{V}) = 0. \tag{14}$$

### *2.3. Momentum Balance*

The method used to establish the equations is the same as in section 2.2, and only the final results are given.

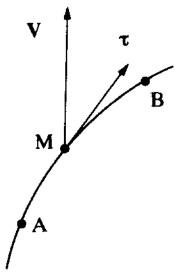

Figure 3. Material line of finite length.

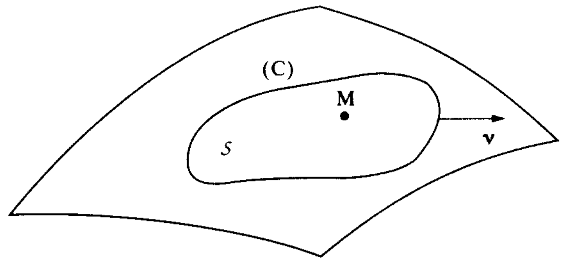

Figure 4. Material surface.

*2.3.1. The fluid line*

The linear momentum balance reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{AB}} \rho_l \mathbf{V}\,\mathrm{d}s = \int_{\mathrm{AB}} \rho_l \mathbf{f}_l\,\mathrm{d}s + \mathbf{T}(B) - \mathbf{T}(A), \qquad [15]$$

where $\mathbf{T}$ is the force exerted on a point of the fluid line by the fluid located on the $\boldsymbol{\tau}$ side and $\mathbf{f}_l$ is the external force per unit length.

It can then be deduced that

$$\frac{\partial}{\partial t}\rho_l \mathbf{V} + (\mathbf{V}\cdot\boldsymbol{\tau})\frac{\partial}{\partial s}(\rho_l \mathbf{V}) + \rho_l \mathbf{V}\left(\boldsymbol{\tau}\cdot\frac{\partial \mathbf{V}}{\partial s}\right) - \rho_l \mathbf{f}_l - \frac{\partial \mathbf{T}}{\partial s} = 0 \qquad [16]$$

or, by introducing the speed $\mathbf{U}_r$,

$$\frac{\partial}{\partial t}\rho_l \mathbf{V} + (\mathbf{U}_r\cdot\boldsymbol{\tau})\frac{\partial}{\partial s}\rho_l \mathbf{V} + \rho_l \mathbf{V}\left(\boldsymbol{\tau}\cdot\frac{\partial \mathbf{U}_r}{\partial s}\right) + \frac{\partial}{\partial s}(\rho_l \mathbf{V}[(\mathbf{V}-\mathbf{U}_r)\cdot\boldsymbol{\tau}]) - \rho_l \mathbf{f}_l - \frac{\partial \mathbf{T}}{\partial s} = 0. \qquad [17]$$

*2.3.2. The fluid sheet*

The linear momentum balance reads:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{S}} \rho_s \mathbf{V}\,\mathrm{d}a = \int_{C} \mathbb{T}_l\cdot\boldsymbol{\nu}\,\mathrm{d}C + \int_{\mathscr{S}} \rho_s \mathbf{f}_s\,\mathrm{d}a, \qquad [18]$$

where $\boldsymbol{\nu}$ is the normal unit vector to $C$ located in the tangential plane to $\mathscr{S}$ and outwardly directed, $\mathbb{T}_l\cdot\boldsymbol{\nu}\,\mathrm{d}C$ is the elementary force exerted on an arc $\mathrm{d}C$ of curve $(C)$ by the fluid located on the $\boldsymbol{\nu}$ side and $\mathbf{f}_s$ is the external force per unit area. Therefore,

$$\frac{\partial}{\partial t}\rho_s \mathbf{V} + \mathrm{div}_s(\rho_s \mathbf{V}\mathbf{V}) - \rho_s \mathbf{f}_s - \mathrm{div}_s\,\mathbb{T}_l^* = 0, \qquad [19]$$

where

$$\mathbb{T}_l^* \triangleq (\mathbb{T}_l\cdot\mathbf{a}^\beta)\mathbf{a}_\beta. \qquad [20]$$

In this relation, the vectors $\mathbf{a}_\beta$ ($\beta = 1, 2$) are the unit vectors of the local basis of the surface. If the speed $\mathbf{U}_s$ is introduced, [19] becomes:

$$\frac{\partial}{\partial t}\rho_s \mathbf{V} + \mathrm{div}_s\,\rho_s(\mathbf{V}\mathbf{U}_s) + \mathrm{div}_s\,\rho_s[\mathbf{V}(\mathbf{V}-\mathbf{U}_s)] - \rho_s \mathbf{f}_s - \mathrm{div}_s\,\mathbb{T}_l^* = 0. \qquad [21]$$

*Remark.* For the two fluid structures studied above, the analogous expressions for the forces exerted by the external parts of the fluid line and the fluid sheet are, respectively:

- fluid line, $\mathbb{T}_P\cdot\boldsymbol{\tau}$ ($P \equiv$ A or B);

and

- fluid sheet, $\mathbb{T}_l\cdot\boldsymbol{\nu}\,\mathrm{d}C$.

The components of the tensors $\mathbb{T}_P$ and $\mathbb{T}_l\,\mathrm{d}C$ have the dimension of a force.

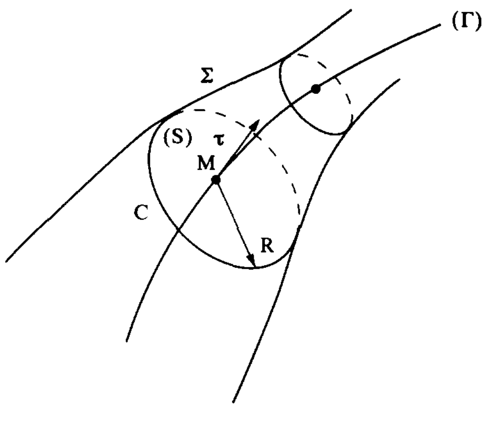


Figure 5. The fluid filament.

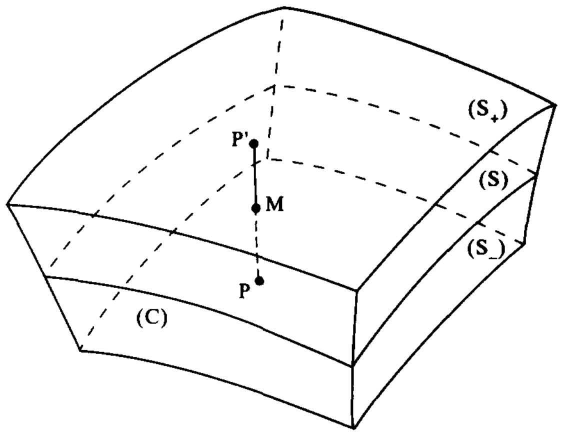


Figure 6. The fluid membrane.

Generally, the force $\mathbb{T}_P \cdot \boldsymbol{\tau}$ is not tangent to the curve $(\Gamma_t)$ and the elementary force $\mathbb{T}_l \cdot \boldsymbol{\nu}\, dC$ is not in the tangential plane to the surface $(S_t)$. We will say that the fluid line is made up of an inviscid fluid if $\mathbb{T}_P = T_P \mathbb{U}$, (i.e. $\mathbb{T}_P \cdot \boldsymbol{\tau} = T_P \boldsymbol{\tau}$) and that the fluid sheet is made up of an inviscid fluid if $\mathbb{T}_l = T_l \mathbb{U}$ (i.e. $\mathbb{T}_l \cdot \boldsymbol{\nu}\, dC = T_l \boldsymbol{\nu}\, dC$), where $\mathbb{U}$ is the unit tensor.

## 3. THE FLUID FILAMENT AND THE FLUID SHEET

### *3.1. Definitions*

#### *3.1.1. The fluid filament*

The fluid filament is the three-dimensional fluid structure associated with the fluid line and is defined in the following way (figure 5). We associate to any point of a space curve $(\Gamma)$ a plane, circular† section $(S)$ of center M, radius $R$ and whose plane is normal to $(\Gamma)$. The radius $R$ depends on the arc length $s$ along $(\Gamma)$ and on the time $t$; we suppose that $R$ is smaller than the radius of curvature $\mathscr{R}$ of $(\Gamma)$.

The volume generated by $(S)$ when M describes $(\Gamma)$ constitutes the *fluid filament*.

$(C)$ is the circumference of radius $R$ whose center is M and which is located in the section plane. $(\Sigma)$ is the lateral surface of the tube defined above and will be called the interface of the filament.

#### *3.1.2. The fluid membrane*

The *fluid membrane* (figure 6) is the three-dimensional fluid structure associated with the fluid sheet and is limited by two surfaces, $(S_+)$ and $(S_-)$, symmetrically located with respect to a surface $(S)$. At any point M of surface $(S)$, one associates a segment PP′, normal to $(S)$ and of length $h$. The two points P and P′ are symmetrically located with respect to M and the length $h$ depends on the time $t$ and M. In the following, a part $(\mathscr{S})$ of the surface $(S)$, limited by the closed curve $(C)$ is considered.

The method used to establish the equations of mass balance and linear momentum balance is the same as for the filament and the membrane. As a result, we will derive in detail only the mass balance for the fluid filament. It is assumed, in the following, that the different quantities are sufficiently smooth.

### *3.2. Mass Balance*

#### *3.2.1. The fluid filament*

We start with the local equation

$$\frac{\partial}{\partial t}\rho_r + \operatorname{div} \rho_r \mathbf{V} = 0, \qquad [22]$$

†For a noncircular surface, the method used leads to analogous results.

where $\rho_r$ is the density of the fluid constituting the filament. We average the local equations over the cross-section area $S(\mathrm{M}, t)$, in the manner suggested in appendix C. We thus write

$$\iint_{S(\mathrm{M},t)} \frac{\partial}{\partial t} \rho_r \lambda \, \mathrm{d}a + \iint_{S(\mathrm{M},t)} \operatorname{div} \rho_r \mathbf{V} \lambda \, \mathrm{d}a = 0,$$

where

$$\lambda \triangleq 1 - \frac{x}{\mathscr{R}}; \qquad [23]$$

$(x, y)$ denote the coordinates of a point located in the cross section whose local basis is $(\mathrm{M}; \mathbf{n}, \mathbf{b})$ and $\mathscr{R}$ is the radius of curvature of $(\Gamma)$. The cross section $S(\mathrm{M}, t)$ is obtained for a fixed value of $p$. By using the Leibniz and Gauss theorems established in appendix C ([C5] and [C7]), the following equation is obtained:

$$\frac{\partial}{\partial t} \iint_{S(\mathrm{M},t)} \rho_r \lambda \, \mathrm{d}a + \left(\frac{\partial \mathbf{U}_\Gamma}{\partial s} \cdot \boldsymbol{\tau}\right) \iint_{S(\mathrm{M},t)} \rho_r \lambda \, \mathrm{d}a + (\mathbf{U}_\Gamma \cdot \boldsymbol{\tau}) \frac{\partial}{\partial s} \iint_{S(\mathrm{M},t)} \rho_r \lambda \, \mathrm{d}a$$
$$+ \frac{\partial}{\partial s} \iint_{S(\mathrm{M},t)} \rho_r (\mathbf{V} - \mathbf{U}_s) \cdot \boldsymbol{\tau} \, \mathrm{d}a + \int_{C(\mathrm{M},t)} \rho_r \frac{(\mathbf{V} - \mathbf{U}_\Sigma) \cdot \mathbf{n}_\Sigma}{\mathbf{n}_\Sigma \cdot \mathbf{n}_C} \lambda \, \mathrm{d}l = 0, \qquad [24]$$

where $\mathbf{U}_\Gamma$ is the speed of a point attached to the geometrical line $(\Gamma)$, $\mathbf{U}_s \cdot \boldsymbol{\tau}$ is the speed of displacement of the cross section $(S)$, $\boldsymbol{\tau}$ is the unit tangent vector to the curve $(\Gamma_t)$, $\mathbf{U}_\Sigma$, $\mathbf{n}_\Sigma$ is the speed of displacement of the interface $(\Sigma)$, $\mathbf{n}_\Sigma$ is the unit vector normal to the interface $(\Sigma)$, outwardly directed from the fluid filament, and $\mathbf{n}_C$ is the unit vector normal to $(C)$ located in the cross-section plane.

Taking into account the definition of an averaged quantity over the section $(S)$,

$$\langle f \rangle \triangleq \frac{\displaystyle\iint_{S(\mathrm{M},t)} f\lambda \, \mathrm{d}a}{\displaystyle\iint_{S(\mathrm{M},t)} \lambda \, \mathrm{d}a} = \frac{\displaystyle\iint_{S(\mathrm{M},t)} f\lambda \, \mathrm{d}a}{\pi R^2}, \qquad [25]$$

[24] can be written as

$$\frac{\partial}{\partial t} \pi R^2 \langle \rho_r \rangle + \pi R^2 \langle \rho_r \rangle \boldsymbol{\tau} \cdot \frac{\partial \mathbf{U}_\Gamma}{\partial s} + (\mathbf{U}_\Gamma \cdot \boldsymbol{\tau}) \frac{\partial}{\partial s} (\pi R^2 \langle \rho_r \rangle)$$
$$+ \frac{\partial}{\partial s} (\pi R^2 \langle \rho_r (\mathbf{V} - \mathbf{U}_s) \cdot \mathbf{g}^1 \rangle) + \int_{C(\mathrm{M},t)} \rho_r \frac{(\mathbf{V} - \mathbf{U}_\Sigma) \cdot \mathbf{n}_\Sigma}{\mathbf{n}_\Sigma \cdot \mathbf{n}_C} \lambda \, \mathrm{d}l = 0, \qquad [26]$$

where $\mathbf{g}^1$ is defined by the relation

$$\mathbf{g}^1 \triangleq \frac{\boldsymbol{\tau}}{\lambda}. \qquad [27]$$

In the absence of phase change at the interface we have

$$(\mathbf{V} - \mathbf{U}_\Sigma) \cdot \mathbf{n}_\Sigma = 0,$$

and the area-averaged mass balance equation now reads:

$$\frac{\partial}{\partial t} \pi R^2 \langle \rho_r \rangle + \pi R^2 \langle \rho_r \rangle \boldsymbol{\tau} \cdot \frac{\partial \mathbf{U}_\Gamma}{\partial s} + (\mathbf{U}_\Gamma \cdot \boldsymbol{\tau}) \frac{\partial}{\partial s} (\pi R^2 \langle \rho_r \rangle) + \frac{\partial}{\partial s} (\pi R^2 \langle \rho_r (\mathbf{V} - \mathbf{U}_s) \cdot \mathbf{g}^1 \rangle) = 0. \qquad [28]$$

*3.2.2. The fluid membrane*

Starting from the local equation

$$\frac{\partial}{\partial t}\rho_r + \operatorname{div}\rho_r \mathbf{V} = 0,$$

we average it over the segment PP′. We then transform this averaged equation by means of the Leibniz theorem, [C12], and the Gauss theorem, [C14], and thus deduce the segment-averaged equation for the mass balance:

$$\frac{\partial}{\partial t}\int_{-h/2}^{+h/2}\mu\rho_r\,\mathrm{d}x^3 + \operatorname{div}_s\mathbf{U}_s\int_{-h/2}^{+h/2}\mu\rho_r\,\mathrm{d}x^3 + \mathbf{U}_s\cdot\mathbf{grad}_s\int_{-h/2}^{+h/2}\mu\rho_r\,\mathrm{d}x^3$$
$$+\operatorname{div}_s\left(\left(\int_{-h/2}^{+h/2}\rho_r\mu(\mathbf{V}-\mathbf{U}_{\Sigma\|})\,\mathrm{d}x^3\right)\right)+\left[\mu\rho_r\frac{(\mathbf{V}-\mathbf{U}_{S_\pm})\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2}=0, \quad [29]$$

where

$$\mu \triangleq 1 - 2Hx^3 + K(x^3)^2 \quad [30]$$

with

$$x^3 = \mathbf{MQ}\cdot\mathbf{a}_3 \quad [31]$$

for every point Q of the segment PP′. $H$ is the mean curvature, $K$ is the Gauss curvature and $\mathbf{a}_3$ is the unit vector normal to the surface $(S)$; $\mathbf{U}_s$ is the speed of a point attached to the surface $(S)$, $\mathbf{U}_{\Sigma\|}$ is the velocity defined by [C11] in appendix C, $\mathbf{U}_{S_\pm}\cdot\mathbf{N}_\pm$ is the speed of displacement of the interface $(S_\pm)$ and $N_\pm$ is the unit vector normal to the interface outwardly directed.

Taking into account the definition of an averaged quantity over the segment PP′,

$$\langle f\rangle \triangleq \frac{\int_{-h/2}^{+h/2} f\mu\,\mathrm{d}x^3}{\int_{-h/2}^{+h/2}\mu\,\mathrm{d}x^3} \quad [32]$$

with

$$h^* \triangleq \int_{-h/2}^{+h/2}\mu\,\mathrm{d}x^3 = h\left(1+K\frac{h^2}{12}\right), \quad [33]$$

the averaged equation can be written as

$$\frac{\partial}{\partial t}h^*\langle\rho_r\rangle + h^*\langle\rho_r\rangle\operatorname{div}_s\mathbf{U}_s + \mathbf{U}_s\cdot\mathbf{grad}_s(h^*\langle\rho_r\rangle)$$
$$+\operatorname{div}_s[h^*\langle\rho_r(\mathbf{V}-\mathbf{U}_{\Sigma\|})\rangle]+\left[\mu\rho_r\frac{(\mathbf{V}_{S_\pm}-\mathbf{U}_{S_\pm})\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2}=0. \quad [34]$$

In the absence of phase change at the interface, the segment-averaged equation for the mass balance can be written as

$$\frac{\partial}{\partial t}h^*\langle\rho_r\rangle + h^*\langle\rho_r\rangle\operatorname{div}_s\mathbf{U}_s + \mathbf{U}_s\cdot\mathbf{grad}_s\,h^*\langle\rho_r\rangle + \operatorname{div}_s(h^*\langle\rho_r(\mathbf{V}-\mathbf{U}_{\Sigma\|})\rangle) = 0. \quad [35]$$

### 3.3. Linear Momentum Balance

#### 3.3.1. The fluid filament

Starting with the local equation of linear momentum, the averaged equation over the section $(S)$ for the momentum balance is found to be

$$\frac{\partial}{\partial t}\pi R^2\{\rho_r \mathbf{V}\} + \pi R^2\{\rho_r \mathbf{V}\}\boldsymbol{\tau}\cdot\frac{\partial \mathbf{U}_\Gamma}{\partial s} + (\mathbf{U}_\Gamma\cdot\boldsymbol{\tau})\frac{\partial}{\partial s}(\pi R^2\{\rho_r\mathbf{V}\})$$

$$+\frac{\partial}{\partial s}\pi R^2\{(\rho_r\mathbf{V}(\mathbf{V}-\mathbf{U}_\Gamma)-\mathbb{T})\cdot\mathbf{g}^1\} - \pi R^2\{\rho_r\mathbf{F}\}$$

$$+\int_C \frac{[\rho_r\mathbf{V}(\mathbf{V}-\mathbf{U}_\Sigma)-\mathbb{T}]\cdot\mathbf{n}_\Sigma}{\mathbf{n}_\Sigma\cdot\mathbf{n}_C}\lambda\,\mathrm{d}l = 0, \qquad [36]$$

where $\mathbf{F}$ is the external force per unit of mass and $\mathbb{T}$ is the stress tensor.

The mass and momentum balances at the interface $(\Sigma)$, when there is no external fluid present, lead to

$$(\mathbf{V}-\mathbf{U}_\Sigma)\cdot\mathbf{n}_\Sigma = 0$$

and

$$\mathbb{T}\cdot\mathbf{n}_\Sigma = 2H_\Sigma\sigma\mathbf{n}_\Sigma,$$

where $\sigma$ is the surface tension and $H_\Sigma$ is the mean curvature of the interface $(\Sigma)$. The averaged equation then becomes

$$\frac{\partial}{\partial t}\pi R^2\{\rho_r \mathbf{V}\} + \pi R^2\{\rho_r \mathbf{V}\}\boldsymbol{\tau}\cdot\frac{\partial \mathbf{U}_\Gamma}{\partial s} + (\mathbf{U}_\Gamma\cdot\boldsymbol{\tau})\frac{\partial}{\partial s}(\pi R^2\{\rho_r\mathbf{V}\})$$

$$+\frac{\partial}{\partial s}\pi R^2\{(\rho_r\mathbf{V}(\mathbf{V}-\mathbf{U}_\Gamma)-\mathbb{T})\cdot\mathbf{g}^1\} - \pi R^2\{\rho_r\mathbf{F}\} - 2\int_C\frac{H_\Sigma\sigma\mathbf{n}_\Sigma}{\mathbf{n}_\Sigma\cdot\mathbf{n}_C}\lambda\,\mathrm{d}l = 0. \qquad [37]$$

#### 3.3.2. The fluid membrane

We start with the local equation of linear momentum. Applying the theorems of Leibniz and Gauss ([C16] and [C18]), we obtain the equation averaged over a segment by means of definition [32]:

$$\frac{\partial}{\partial t}h^*\langle\rho_r\mathbf{V}\rangle + h^*\langle\rho_r\mathbf{V}\rangle\mathrm{div}_s\,\mathbf{U}_s + \mathbf{grad}_s(h^*\langle\rho_r\mathbf{V}\rangle)\cdot\mathbf{U}_s$$

$$+\mathrm{div}_s[h^*\langle(\rho_r\mathbf{V}(\mathbf{V}-\mathbf{U}_\Sigma)-\mathbb{T})\cdot\mathbf{g}^\beta\rangle\mathbf{a}_\beta]$$

$$+\left[\frac{\mu[\rho_r\mathbf{V}(\mathbf{V}-\mathbf{U}_{S_\pm})-\mathbb{T}]\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2} - h^*\langle\rho_r\mathbf{F}\rangle = 0. \qquad [38]$$

In [38] $\mathbf{a}_\beta$ $(\beta = 1, 2)$ denote the base vectors of the local basis of the surface $(S)$ $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ is the local basis of the space associated with the curvilinear coordinates system $x^\alpha$, $x^3$. The mass and momentum balances at the interface $(\Sigma)$, if there is no external fluid present, lead to

$$(\mathbf{V}-\mathbf{U}_{S_\pm})\cdot\mathbf{N}_\pm = 0$$

and

$$\mathbb{T}\cdot\mathbf{N}_\pm = 2H_\pm\sigma\mathbf{N}_\pm,$$

where $H_\pm$ denotes the mean curvature of the interface $S_\pm$. The averaged equation can then be written as

$$\frac{\partial}{\partial t}h^*\langle\rho_r\mathbf{V}\rangle + (\mathrm{div}_s\,\mathbf{U}_s)h^*\langle\rho_r\mathbf{V}\rangle + \mathbf{grad}_s\,h^*\langle\rho_r\mathbf{V}\rangle\cdot\mathbf{U}_s$$

$$+\,\mathrm{div}_s[h^*\langle(\rho_r\mathbf{V}(\mathbf{V}-\mathbf{U}_\Sigma)-\mathbb{T})\cdot\mathbf{g}^\beta\rangle\mathbf{a}_\beta] - h^*\langle\rho_r\mathbf{F}\rangle - \left[\frac{2H_\pm\sigma\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\mu\right]_{-h/2}^{+h/2} = 0. \quad [39]$$

## 4. THE THIN FILAMENT AND THE THIN MEMBRANE

### 4.1. *Introduction*

We consider now a thin filament or a thin membrane. The filament, as in section 3, is built on a curve $(\Gamma_t)$ and is generated by a disk of radius $R(s,t)$ normal to $\Gamma_t$. The diameter of the cross section remains much smaller than a characteristic length of the line $(\Gamma_t)$. For a rectilinear filament, a characteristic length can be the length of the filament. For a curvilinear filament, a characteristic length can be chosen as

$$\mathscr{R}_{\min} = \inf(\mathscr{R}, \mathscr{T}),$$

where $1/\mathscr{R}$ is the curvature and $1/\mathscr{T}$ is the torsion. Similarly, the membrane is built on a surface $(S)$ and is generated by segments orthogonal to the surface $S$, symmetric with respect to $(S)$. The half-thickness of the membrane remains smaller than a characteristic length of the surface $S$.

For a plane membrane, a characteristic length can be the diameter. For any surface, a characteristic length can be chosen as

$$R = \inf(R_1, R_2),$$

where $R_1$ and $R_2$ are the principal radii of curvature of the surface $S$.

The reader will find some geometrical properties of these domains in appendix D.

Two different models of thin fluid structures can be developed depending on the problem to handle and the desired degree of accuracy:

(1) The first model is a one-dimensional model based on a one-dimensional approximation. The balance equations are obtained from the equations of section 2, where the line (or surface) density is replaced by the product of a constant volumetric density by a cross-section area (or a height).

(2) The second model is a one-dimensional model using the averaged balance equations. The balance equations are obtained by asymptotically reducing the thickness of the filament or the membrane in the equations derived in section 3.

The balance equations for both models will be derived for the fluid membrane. The corresponding equations for the fluid filament will be given in table 1 at the end of this section.

### 4.2. *One-dimensional Model Based on a One-dimensional Approximation*

#### 4.2.1. *Mass balance*

The relation between the mass per unit area $\rho_s$ and the density of the fluid $\rho_r$ can be written as (appendix D, [D6]):

$$\rho_s = \rho_r\int_{-h/2}^{+h/2}\mu\,\mathrm{d}x^3$$

or by means of [30],

$$\rho_s = \rho_r h^* = \rho_r h\left(1 + K\frac{h^2}{12}\right) \simeq \rho_r h,$$

where the hypothesis on the thickness of the membrane given in section 4.1 has been taken into account. Consequently, [14] and [12] become

$$\frac{\partial}{\partial t} h + \operatorname{div}_s h\mathbf{V} = 0 \qquad [40]$$

and

$$\frac{\partial}{\partial t} h + \operatorname{div}_s h\mathbf{U}_s + \operatorname{div}_s h(\mathbf{V} - \mathbf{U}_s)^* = 0. \qquad [41]$$

An equation similar to [41] has been obtained by Zak (1979). Actually the second term of [41] is missing in equation [1.9] of Zak's paper.

*4.2.2. Linear momentum balance*

By means of the relation $\rho_s = \rho_t h$, [19] becomes

$$\frac{\partial}{\partial t}(h\mathbf{V}) + \operatorname{div}_s(h\mathbf{V}\mathbf{V}) - h\mathbf{f}_s - \frac{1}{\rho_t}\operatorname{div}_s \mathbb{T}_l^* = 0, \qquad [42]$$

where $\mathbb{T}_l^* \triangleq (\mathbb{T}_l \cdot \mathbf{a}^\beta)\mathbf{a}_\beta$.

We deduce from [21]:

$$\frac{\partial}{\partial t}(h\mathbf{V}) + \operatorname{div}_s h\mathbf{V}\mathbf{U}_s + \operatorname{div}_s h[\mathbf{V}(\mathbf{V} - \mathbf{U}_s)] - h\mathbf{f}_s - \frac{1}{\rho_t}\operatorname{div}_s \mathbb{T}_l^* = 0. \qquad [43]$$

Taking into account [40] and [41], [42] and [43] become

$$\frac{\partial \mathbf{V}}{\partial t} + \operatorname{grad}_s \mathbf{V} \cdot \mathbf{V} - \mathbf{f}_s - \frac{1}{\rho_t h}\operatorname{div}_s \mathbb{T}_l^* = 0 \qquad [44]$$

and

$$\frac{\partial \mathbf{V}}{\partial t} + \operatorname{grad}_s \mathbf{U}_s \cdot \mathbf{V} + \operatorname{grad}_s(\mathbf{V} - \mathbf{U}_s) \cdot \mathbf{V} - \mathbf{f}_s - \frac{1}{\rho_t h}\operatorname{div}_s \mathbb{T}_l^* = 0. \qquad [45]$$

As for the mass balance, the equation given by Zak (1979) is incomplete due to the acceleration term. Furthermore, Zak considered only the case where

$$\mathbb{T} = T_0 \mathbb{1},$$

where $\mathbb{1}$ is the unit metric tensor associated with the surface $(S)$ and $T_0$ is assumed constant.

*Remark*. If the fluid of the fluid sheet is inviscid, then

$$\mathbb{T}_l = T_l \mathbb{U}$$

and [44] becomes

$$\frac{\partial \mathbf{V}}{\partial t} + \operatorname{grad}_s \mathbf{V} \cdot \mathbf{V} - \mathbf{f}_s - \frac{1}{\rho_t h}\,\mathbf{grad}_s\, T_l - \frac{2HT_l}{\rho_t h}\mathbf{a}_3 = 0. \qquad [46]$$

### *4.3. One-dimensional Model Using the Averaged Balance Equations*

*4.3.1. Mass balance*

A small parameter $\epsilon$ is defined as

$$\epsilon \triangleq \frac{h}{R} \quad \text{where} \quad R = \inf(R_1, R_2), \qquad [47]$$

then the terms of [29] and [34] are developed with respect to $\epsilon$. One keeps only the terms of order 0 in $\epsilon$.

The averaging operator, in this case, denoted by $\langle f\rangle^{\bullet}$ is the classical segment averaging operator:

$$\langle f\rangle^{\bullet}=\frac{\int_{-h/2}^{+h/2} f\,\mathrm{d}x^3}{\int_{-h/2}^{+h/2}\mathrm{d}x^3}=\frac{\int_{-h/2}^{+h/2} f\,\mathrm{d}x^3}{h}. \qquad [48]$$

Equation [34] thus becomes

$$\frac{\partial}{\partial t}h\langle\rho_v\rangle^{\bullet}+h\langle\rho_v\rangle^{\bullet}\operatorname{div}_s\mathbf{U}_s+\mathbf{U}_s\cdot\mathbf{grad}_s(h\langle\rho_v\rangle^{\bullet})$$
$$+\operatorname{div}_s[h\langle\rho_v(\mathbf{V}-\mathbf{U}_\Sigma)^{*}\rangle^{\bullet}]+\left[\frac{\rho_v(\mathbf{V}_{S\pm}-\mathbf{U}_{S\pm})\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2}=0. \qquad [49]$$

For any point Q of the segment PP′, we have:

$$(\mathbf{V}-\mathbf{U}_\Sigma)^{*}_{\mathrm{Q}}=(\mathbf{V}-\mathbf{U}_\Sigma)^{*}_{\mathrm{M}}=(\mathbf{V}_{\mathrm{M}}-\mathbf{U}_{\mathrm{M}})^{*}=\mathbf{V}-\mathbf{U}_s.$$

Equation [49] can be written as

$$\frac{\partial}{\partial t}h\langle\rho_v\rangle^{\bullet}+\operatorname{div}_s(h\mathbf{V}\langle\rho_v\rangle^{\bullet})+\left[\frac{\rho_v(\mathbf{V}_{S\pm}-\mathbf{U}_{S\pm})\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2}=0. \qquad [50]$$

For a constant density $\rho_v$ one obtains:

$$\frac{\partial}{\partial t}h+\operatorname{div}_s(h\mathbf{V})+\left[\frac{\rho_v(\mathbf{V}_{S\pm}-\mathbf{U}_{S\pm})\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2}=0. \qquad [51]$$

In the absence of phase change at the interface, [42], in section 4, is recovered:

$$\frac{\partial}{\partial t}h+\operatorname{div}_s(h\mathbf{V})=0. \qquad [52]$$

Taking into account the hypothesis $h/R \ll 1$, [52] can also be written as

$$\frac{\partial}{\partial t}h+\operatorname{div}_s(h\langle\mathbf{V}\rangle^{\bullet})=0. \qquad [53]$$

*4.3.2. Linear momentum balance*

Taking into account the hypotheses made in section 4.1, [38] becomes

$$\frac{\partial}{\partial t}h\langle\rho_v\mathbf{V}\rangle^{\bullet}+h\langle\rho_v\mathbf{V}\rangle^{\bullet}\operatorname{div}_s\mathbf{U}_s+\operatorname{grad}_s h\langle\rho_v\mathbf{V}\rangle^{\bullet}\cdot\mathbf{U}_s$$
$$+\operatorname{div}_s[h\langle(\rho_v\mathbf{V}(\mathbf{V}-\mathbf{U}_\Sigma)-\mathbb{T})\cdot\mathbf{a}^{\beta}\rangle^{\bullet}\mathbf{a}_\beta]$$
$$+\left[\frac{(\rho_v\mathbf{V}(\mathbf{V}-\mathbf{U}_{S\pm})-\mathbb{T})\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2}-h\langle\rho_v\mathbf{F}\rangle^{\bullet}=0. \qquad [54]$$

However, at any point Q of the segment PP′:

$$(\mathbf{V}-\mathbf{U}_\Sigma)^{*}_{\mathrm{Q}}=\mathbf{V}-\mathbf{U}_s.$$

If one assumes that $\rho_v$ is constant and if there is no external fluid, [54] becomes

$$\frac{\partial}{\partial t}h\langle\mathbf{V}\rangle^{\bullet}+\operatorname{div}_s\{[(h\langle\mathbf{V}\rangle^{\bullet}\langle\mathbf{V}\rangle^{\bullet})\cdot\mathbf{a}^{\beta}]\mathbf{a}_\beta\}$$
$$-\frac{1}{\rho_v}\operatorname{div}_s\left\{\left(\int_{-h/2}^{+h/2}\mathbb{T}\cdot\mathbf{a}^{\beta}\,\mathrm{d}x^3\right)\mathbf{a}_\beta\right\}-h\langle\mathbf{F}\rangle^{\bullet}-\frac{1}{\rho_v}\left[\frac{2H_\pm\sigma\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2}=0. \qquad [55]$$

Equation [55] is transformed by means of the mass balance and leads to

$$\frac{\partial}{\partial t}\langle \mathbf{V}\rangle^{\bullet} + \mathrm{grad}_s\langle \mathbf{V}\rangle^{\bullet}\cdot\langle \mathbf{V}\rangle^{\bullet} - \frac{1}{\rho_r h}\mathrm{div}_s(h\langle \mathbb{T}^*\rangle^{\bullet}) - \langle \mathbf{F}\rangle^{\bullet} - \frac{1}{\rho_r h}\left[\frac{2H_\pm \sigma \mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2} = 0. \qquad [56]$$

As a result, [44] is recovered, where

$\langle \mathbf{V}\rangle^{\bullet}$ replaces $\mathbf{V}$,

$\langle \mathbf{F}\rangle^{\bullet}$ replaces $\mathbf{f}_s$

and

$$\mathrm{div}_s\, h\langle \mathbb{T}^*\rangle^{\bullet} + \sigma\left[\frac{2H_\pm \mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2} \quad \text{replaces} \quad \mathrm{div}_s\,\mathbb{T}_l^*$$

Table 1. Balance equations for the thin fluid structures

| | | First model | Second model |
|---|---|---|---|
| FLUID FILAMENT | Mass balance | $\frac{\partial}{\partial t}R^2 + \boldsymbol{\tau}\cdot\frac{\partial}{\partial s}(R^2\mathbf{V}) = 0$ | $\frac{\partial}{\partial t}R^2 + \boldsymbol{\tau}\cdot\frac{\partial}{\partial s}(R^2\{\mathbf{V}\}^{\bullet}) = 0$ |
| | Linear momentum balance | $\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\tau})\frac{\partial \mathbf{V}}{\partial s} - \mathbf{f}_l - \frac{1}{\rho_l}\frac{\partial \mathbf{T}}{\partial s} = 0$ | $\frac{\partial}{\partial t}\{\mathbf{V}\}^{\bullet} + (\{\mathbf{V}\}^{\bullet}\cdot\boldsymbol{\tau})\frac{\partial}{\partial s}\{\mathbf{V}\}^{\bullet} - \{\mathbf{F}\}^{\bullet} - \frac{1}{\rho_r R^2}\frac{\partial}{\partial s}(R^2\{\mathbb{T}\cdot\boldsymbol{\tau}\|^{\bullet}) - \frac{1}{\pi R^2\rho_r}\int_C \frac{2H_\Sigma\sigma\mathbf{n}_\Sigma}{\mathbf{n}_\Sigma\cdot\mathbf{n}_C}\,\mathrm{d}l = 0$ |
| FLUID MEMBRANE | Mass balance | $\frac{\partial h}{\partial t} + \mathrm{div}_s(h\mathbf{V}) = 0$ | $\frac{\partial h}{\partial t} + \mathrm{div}_s(h\langle\mathbf{V}\rangle^{\bullet}) = 0$ |
| | Linear momentum balance | $\frac{\partial \mathbf{V}}{\partial t} + \mathrm{grad}_s\mathbf{V}\cdot\mathbf{V} - \mathbf{f}_s - \frac{1}{\rho_s}\mathrm{div}_s\,\mathbb{T}_l^* = 0$ | $\frac{\partial}{\partial t}\langle\mathbf{V}\rangle^{\bullet} + \mathrm{grad}_s\langle\mathbf{V}\rangle^{\bullet}\cdot\langle\mathbf{V}\rangle^{\bullet} - \langle\mathbf{F}\rangle^{\bullet} - \frac{1}{\rho_r h}\mathrm{div}_s(h\langle\mathbb{T}^*\rangle^{\bullet}) - \frac{1}{\rho_r h}\left[\frac{2H_\pm\sigma\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2} = 0$ |

## 5. SOME TYPICAL APPLICATIONS

### *5.1. One-dimensional Equations for Single-phase Flow in Bends*

One-dimensional modeling of single-phase flow in a straight pipe is based on area-averaged equations as derived by Delhaye (1981). This formulation could be easily extended to single-phase flow in bends by just changing the rectilinear coordinate into the curvilinear coordinate $s$ along the axis of the bend. This procedure would lead to the following equations:

*mass balance*,

$$\frac{\partial}{\partial t}\iint_S \rho_r\,\mathrm{d}a + \frac{\partial}{\partial s}\iint_S \rho_r V_\tau\,\mathrm{d}a = 0; \qquad [57]$$

--- (1) ---

and

*momentum balance*,

$$\frac{\partial}{\partial t}\iint_S \rho_v V_\tau \, \mathrm{d}a + \frac{\partial}{\partial s}\iint_S \rho_v V_\tau^2 \, \mathrm{d}a - \iint_S \rho_v F_\tau \, \mathrm{d}a$$

- - - (2) - - - - - - (3) - - -

$$+\frac{\partial}{\partial s}\iint_S p \, \mathrm{d}a - \frac{\partial}{\partial s}\iint_S (\boldsymbol{\tau} \cdot \mathbb{T}_\mathrm{v}) \cdot \boldsymbol{\tau} \, \mathrm{d}a$$

$$= \int_C \boldsymbol{\tau} \cdot (\mathbb{T} \cdot \mathbf{n}_\Sigma) \, \mathrm{d}C; \qquad [58]$$

- - - (4) - - -

where $V_\tau$ and $V_\mathrm{n}$ are the tangential and normal velocity components, $F_\tau$ is the tangential component of the applied force $\mathbf{F}$, $p$ is the pressure and $\mathbb{T}_\mathrm{v}$ is the viscous stress tensor.

Actually, [57] and [58] are erroneous because they do not reflect the influence of the bend curvature. The correct equations are obtained from [26] and [36] in section 3 and read:

*mass balance*,

$$\frac{\partial}{\partial t}\iint_S \rho_v \lambda \, \mathrm{d}a + \frac{\partial}{\partial s}\iint_S \rho_v V_\tau \, \mathrm{d}a; \qquad [59]$$

- - - (1′) - - -

and

*momentum balance* (*tangential projection*),

$$\frac{\partial}{\partial t}\iint_S \rho_v V_\tau \lambda \, \mathrm{d}a + \frac{\partial}{\partial s}\iint_S \rho_v V_\tau^2 \, \mathrm{d}a - \iint_S \rho_v F_\tau \lambda \, \mathrm{d}a$$

- - - (2′) - - - - - - (3′) - - -

$$+\frac{\partial}{\partial s}\iint_S p \, \mathrm{d}a - \frac{\partial}{\partial s}\iint_S (\boldsymbol{\tau} \cdot \mathbb{T}_\mathrm{v}) \cdot \boldsymbol{\tau} \, \mathrm{d}a$$

$$-\iint_S \rho_v V_\tau \frac{V_\mathrm{n}}{\mathscr{R}} \mathrm{d}a - \iint_S (\boldsymbol{\tau} \cdot \mathbb{T}_\mathrm{v}) \cdot \frac{\mathbf{n}}{\mathscr{R}} \mathrm{d}a = \int_C \boldsymbol{\tau} \cdot (\mathbb{T} \cdot \mathbf{n}_\Sigma) \lambda \, \mathrm{d}C. \qquad [60]$$

- - - (5′) - - - - - - (6′) - - - - - - (4′) - - -

Some terms are identical in the two systems but terms (1)–(4) must be replaced by terms (1′)–(4′) and two new terms (5′) and (6′) appear in [59] and [60]. These new terms (5′) and (6′) involve the normal components of the velocity and of the stress tensor.

Equations [59] and [60] must obviously be complemented by the following projection of the momentum equation along the normal direction $\mathbf{n}$ to the axis of the bend:

$$\frac{\partial}{\partial t}\iint_S \rho_r V_n \lambda \, da + \frac{\partial}{\partial s}\iint_S \rho_r V_n V_\tau \, da - \iint_S \rho_r F_n \lambda \, da$$
$$-\frac{\partial}{\partial s}\iint_S \mathbf{n}\cdot(\mathbb{T}_v \cdot \boldsymbol{\tau})\, da + \iint_S \rho_r \frac{V_\tau^2}{\mathscr{R}}\, da + \iint_S \frac{p}{\mathscr{R}}\, da$$
$$-\iint_S \frac{1}{\mathscr{R}}\boldsymbol{\tau}\cdot(\mathbb{T}_v \cdot \boldsymbol{\tau})\, da = \int_C \mathbf{n}\cdot(\mathbb{T}\cdot \mathbf{n}_\Sigma)\lambda \, dC. \qquad [61]$$

In a practical case, a scale analysis of [59]–[61] should be performed to specify the order of magnitude of the extra terms or coefficients.

### 5.2. *Dynamic Centering of Thin Liquid Shells in Capillary Oscillations*

We consider compound drops which consist of a gas domain enclosed within a thin fluid shell immersed in an outside gas. Here the inner gas is the same as the outer one. It was observed in many experiments that a compound drop in oscillation tends to become concentric.

The averaged equations established in section 4 can be used to study the motion of such compound drop under the following assumptions:

- The thickness of the liquid shell is much smaller than its characteristic dimension.
- The internal flow in the liquid shell is ignored.
- The inner gas is incompressible and the external pressure is uniform.
- The liquid shell is subjected only to the effects of surface tension and of internal and external gaseous pressures.

The inner and outer surfaces of the axisymmetric thin shell we consider are defined by their position vectors $R_i$ and $R_o$, respectively. The mid-surface is characterized by the spherical polar coordinate $\theta$, $R \triangleq \frac{1}{2}(R_i + R_o)$. The velocity field for points located on the mid-surface is defined by its tangential and normal components $V_s$ and $V_n$. The unknowns of this problem are the radius, the equivalent surface density $\rho_S$, $V_s$ and $V_n$.

The mass balance equation, the momentum balance equation and the shell displacement velocity can be obtained directly from [50] and [56] of section 4 and read:

$$\frac{\partial \rho_S}{\partial t} = \frac{K}{R}\frac{\partial \rho_S}{\partial \theta} - \frac{\rho_S}{R}\left[\cos\psi \frac{\partial V_s}{\partial \theta} + f_r \cos\psi + V_s \sin\psi\right] - \rho_S C V_n, \qquad [62]$$

$$\frac{\partial V_s}{\partial t} = -\frac{K}{R}\frac{\partial V_s}{\partial \theta} - V_n H - \frac{\cos\psi}{\rho_S R}\frac{\partial p}{\partial \theta}, \qquad [63]$$

$$\frac{\partial V_n}{\partial t} = -\frac{K}{R}\frac{\partial V_n}{\partial \theta} + V_s H + \frac{1}{\rho_S}(P + \Delta p) \qquad [64]$$

and

$$V_n = \cos\psi \frac{\partial R}{\partial t}, \qquad [65]$$

where

$$\tan\psi \triangleq \frac{1}{R}\frac{\partial R}{\partial \theta}, \qquad [66]$$

$$K \triangleq V_s \cos\psi, \qquad [67]$$

$$H \triangleq -\frac{1}{R}\frac{\partial V_n}{\partial \theta}\cos\psi + \frac{1}{R}(1+\sin^2\psi)(V_n \sin\psi - V_s\cos\psi)$$
$$-\frac{1}{R^2}\cos^2\psi(V_n\sin\psi - V_s\cos\psi)\frac{\partial^2 R}{\partial\theta^2}, \qquad [68]$$

$$f_v \triangleq \begin{cases} V_s\dfrac{\cos\theta}{\sin\theta} & 0<\theta<\pi \\ \dfrac{\partial V_s}{\partial\theta} & \theta = 0 \text{ or } \pi, \end{cases} \qquad [69]$$

$$C \triangleq \frac{1}{R}\left\{2\cos\psi + \sin^2\psi\cos\psi - f - \frac{\cos^3\psi}{R}\frac{\partial^2 R}{\partial\theta^2}\right) \qquad [70]$$

and

$$f \triangleq \begin{cases} \sin\psi\dfrac{\cos\theta}{\sin\theta} & 0<\theta<\pi \\ \dfrac{1}{R}\dfrac{\partial^2 R}{\partial\theta^2} & \theta = 0 \text{ or } \pi; \end{cases} \qquad [71]$$

$C$ is the curvature of the shell, $P$ is the inner pressure, $p$ is the mean liquid pressure within the shell and $\Delta p$ is the difference of pressures between the two sides of the shell. The inner pressure $P$, $p$ and $\Delta p$ are evaluated in terms of interfacial tension, geometrical quantities and $V_n$, $V_s$ and $\rho_S$.

The same problem was studied by Lee & Wang (1988) but their [3.11], [3.19] and [3.20] are incorrect and should be replaced by [62], [63] and [64], respectively. The error comes from their expressions for $K$ and $H$, which are erroneous. According to Lee & Wang, $K$ and $H$ read:

$$K_{LW} \triangleq V_s\cos\psi - V_n\sin\psi \qquad [72]$$

and

$$H_{LW} \triangleq \frac{K_{LW}}{R} - \frac{\cos\psi}{R}\frac{\partial V_n}{\partial\theta} + \frac{\cos\psi\sin\psi}{R}(V_s\sin\psi + V_n\cos\psi) - V_s\frac{\cos^2\psi}{R^2}\frac{\partial^2 R}{\partial\theta^2} + V_n\frac{\sin^3\psi}{R}. \qquad [73]$$

The expressions $K_{LW}$ and $H_{LW}$ for $K$ and $H$ should be replaced by [67] and [68] above.

The problem of the dynamic centering of thin liquid shells in capillary oscillations is currently being revisited with the correct physical equations (Coutris 1993a).

### *5.3. Annular Jet Instability*

We consider a hollow liquid jet formed by an annular nozzle. The liquid jet is a cylindrical liquid sheet enclosing a gas stream. The jet breaks up downstream of the nozzle forming hollow capsule-like liquid shells.

The averaged equations established in section 4 can also be used to study the motion of the liquid sheet if it is assumed that:

- The liquid layer is thin and its thickness is much smaller than its characteristic length.
- The internal flow in the sheet is ignored.
- The gas inside the sheet is incompressible.
- The liquid shell is subjected only to the effects of surface tension and of internal and external gas pressures.

The radial position of the liquid sheet is described by the cylindrical coordinates $z$ and $R$. The velocity field for points on the sheet is defined by its tangential and normal components

$V_s$ and $V_n$. The balance equations and the definition of the shell displacement velocity can be written:

$$\frac{\partial m}{\partial t} = -A\frac{\partial m}{\partial z} - m\left(\cos\theta\frac{\partial V_s}{\partial z} - V_n\frac{\partial^2 R}{\partial z^2}\cos^2\theta\right), \quad [74]$$

$$\frac{\partial V_s}{\partial t} = -A\frac{\partial V_s}{\partial z} - HV_n, \quad [75]$$

$$\frac{\partial V_n}{\partial t} = -A\frac{\partial v_n}{\partial z} + HV_s + \frac{R}{m}\left[P - 2\sigma\left(\frac{\cos\theta}{R} - \frac{\partial^2 R}{\partial z^2}\cos^3\theta\right)\right] \quad [76]$$

and

$$V_n = \cos\theta\frac{\partial R}{\partial t}, \quad [77]$$

where

$$\tan\theta \triangleq \frac{\partial R}{\partial z}, \quad [78]$$

$$m \triangleq \rho_S R, \quad [79]$$

$$A \triangleq V_s\cos\theta, \quad [80]$$

$$H \triangleq \cos\theta\left(\frac{\partial V_n}{\partial z} + \cos^2\theta\frac{\partial^2 R}{\partial z^2}V_s + \cos\theta\sin\theta\frac{\partial^2 R}{\partial z^2}V_n\right), \quad [81]$$

$P$—the pressure difference between the inside and outside of the liquid shell

and

$\sigma$—the interfacial tension.

The same problem was studied by Lee & Wang (1986, 1989) but the equations proposed by these authors are incorrect. Equations [19], [27] and [28] of Lee & Wang (1986) should be replaced by [74], [75] and [76], respectively. The error comes from their expression for $A$ and $H$, which are erroneous. According to Lee & Wang, $A$ and $H$ read:

$$A_{LW} = \cos\theta\left(V_s - V_n\frac{\partial R}{\partial z}\right) \quad [82]$$

and

$$H_{LW} = \cos\theta\left(\frac{\partial V_n}{\partial z} + V_s\frac{\partial^2 R}{\partial z^2}\cos^2\theta\right). \quad [83]$$

The expressions $A_{LW}$ and $H_{LW}$ for $A$ and $H$ should be replaced by [80] and [81] above.

The problem of the annular jet instability is currently being revisited with the correct physical equations (Coutris 1993b).

### *5.4. Dynamics of a Flame Front*

In Candel & Poinsot (1990), an expression for the flame stretch $\phi_S$ is derived. With our notation, it reads:

$$\phi_S = \mathrm{div}_S\,\mathbf{w}, \quad [84]$$

where $\mathbf{w}$ is the velocity of the flame front.

If the front propagates in the normal direction at a speed $S_L$, $\mathbf{w}$ is the sum of two contributions: the local fluid velocity $\mathbf{v}$ and the flame speed in the normal direction $S_L\mathbf{n}$. As a result $\phi_S$ can also be written as

$$\phi_S = \mathrm{div}_S\mathbf{v} + S_L\,\mathrm{div}_S\mathbf{n}. \quad [85]$$

The derivation of a balance equation for the flame area per unit mass, denoted by $a_f$, is straightforward with our method.

Following section 2, the mass balance equation for a flame front is

$$\frac{\partial}{\partial t}\rho_S + \mathrm{div}_S(\rho_S \mathbf{w}) = 0, \qquad [86]$$

where $\rho_S$ is the density of the flame front. It can easily be deduced that, for $a_f \equiv 1/\rho_S$:

$$\frac{\partial}{\partial t} a_f + \mathbf{v} \cdot \mathbf{grad}_S\, a_f = a_f \phi_S. \qquad [87]$$

If the flame area per unit volume is considered, the balance equation for the flame area can be written as

$$\frac{\partial}{\partial t}(\rho_v a_f) + \mathrm{div}_S[\rho a_f(\mathbf{v} + S_L \mathbf{n})] + \frac{\partial}{\partial n}(\rho a_f S_L) = a_f \phi_S. \qquad [88]$$

This is another form of [43] of Candel & Poinsot (1990).

## 6. CONCLUSION

The mass and momentum balances for fluid structures such as lines, filaments, sheets or membranes have been derived using original theorems of the Leibniz and Gauss type. The demonstrations of these theorems are given in the appendices.

The consistency of the formulations has been checked and verified by looking at the asymptotic forms of the balance equations for the three-dimensional structures.

Finally, examples were given for which the use of such balance equations is appropriate.

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# APPENDIX A

*Time Derivatives of Line Integrals*

Let $C(t)$ be a *moving geometrical line*. We denote by $\mathrm{U}(t)$ the velocity of a point on the line. Our objective is to evaluate the derivative of the line integral $I$, defined by

$$I \triangleq \int_{\mathrm{AB}} g(\mathrm{M}, t)\, \mathrm{d}s$$

where AB is an arc on the line $C(t)$ and $g$ is a function defined on this line.

At time $t$, every point on the line is determined by

$$\mathbf{OM} = \mathbf{OM}(p, t) = \sum_{i=1}^{3} x_i(p, t)\mathbf{e}_i$$

where $p$ is a parameter and $x_i$ $(i = 1, 2, 3)$ are the Cartesian coordinates of M with respect to the basis $\{\mathbf{e}_i\}$.

At time $t = 0$, let $a_i$ $(i = 1, 2, 3)$ denote the coordinates of the corresponding point $\mathrm{M}_0$ whose parameter is $p_0$:

$$\mathbf{OM}_0 = \mathbf{OM}(p_0, 0) = \sum_{i=1}^{3} a_i(p_0)\mathbf{e}_i,$$

where

$$p = p(p_0, t).$$

The arc length is denoted by $s$ for the curve $C(t)$ and $s_0$ for the curve $C_0$, and is given by

$$s = s(p, t) = s[p(p_0, t), t],$$

with

$$s_0 = s(p_0, 0).$$

At time $t$, the element of arc $\mathrm{d}s$ is given by

$$\mathrm{d}s = \frac{\partial s}{\partial p}\, \mathrm{d}p.$$

The line integral $I$ can be written as

$$I = \int_{\mathrm{AB}} g(\mathrm{M}, t)\, \mathrm{d}s = \int_{\mathrm{AB}} g(\mathrm{M}, t)\frac{\partial s}{\partial p}\, \mathrm{d}p.$$

By introducing the initial curve $C_0$, we obtain:

$$I = \int_{\mathrm{A}_0\mathrm{B}_0} g[\mathrm{M}(\mathrm{M}_0, t), t]\frac{\partial s}{\partial p}\frac{\partial p}{\partial p_0}\mathrm{d}p_0,$$

hence

$$I = \int_{\mathrm{A}_0\mathrm{B}_0} G(\mathrm{M}_0, t)\lambda\, \mathrm{d}p_0$$

with

$$G(\mathrm{M}_0, t) \triangleq g[\mathrm{M}(\mathrm{M}_0, t), t]$$

and

$$\lambda \triangleq \frac{\partial s}{\partial p}\frac{\partial p}{\partial p_0} = \frac{\partial s}{\partial p_0}.$$

Therefore,

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{AB}} g\,\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{A_0B_0}} G\lambda\,\mathrm{d}p_0$$

and

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\int_{\mathrm{A_0B_0}}\frac{\partial}{\partial t}(G\lambda)\,\mathrm{d}p_0$$

$$=\int_{\mathrm{A_0B_0}}\frac{\partial G}{\partial t}\lambda\,\mathrm{d}p_0+\int_{\mathrm{A_0B_0}} G\frac{\partial\lambda}{\partial t}\,\mathrm{d}p_0.$$

The first term on the LHS can be written as

$$\int_{\mathrm{A_0B_0}}\frac{\partial G}{\partial t}\lambda\,\mathrm{d}p_0=\int_{\mathrm{AB}}\frac{\mathrm{d}g}{\mathrm{d}t}\,\mathrm{d}s.$$

The second term on the LHS requires the calculation of $\partial\lambda/\partial t$. We have:

$$\lambda\frac{\partial\lambda}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\left[\left(\frac{\partial s}{\partial p_0}\right)^2\right]$$

$$=\frac{1}{2}\frac{\partial}{\partial t}\left[\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial p_0}\right)^2\right]$$

$$=\sum_{i=1}^{3}\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial p_0}\right)\frac{\partial x_i}{\partial p_0}$$

$$=\sum_{i=1}^{3}\frac{\partial}{\partial p_0}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\right)\frac{\partial x_i}{\partial p_0}$$

$$=\frac{\partial\mathbf{U}}{\partial p_0}\cdot\frac{\partial\mathbf{OM}}{\partial p_0},$$

where $\mathbf{U}(t)$ is the velocity of a point on the line $C(t)$. The components $U_i$ are such that

$$U_i(x_j,t)=U_i[s(p,t),t]$$
$$=U_i\{s[p(p_0,t),t]\}.$$

Therefore,

$$\frac{\partial\mathbf{U}}{\partial p_0}=\frac{\partial\mathbf{U}}{\partial s}\frac{\partial s}{\partial p_0}$$

and

$$\frac{\partial\mathbf{U}}{\partial p_0}\cdot\frac{\partial\mathbf{OM}}{\partial p_0}=\frac{\partial\mathbf{U}}{\partial s}\cdot\frac{\partial\mathbf{OM}}{\partial s}\left(\frac{\partial s}{\partial p_0}\right)^2,$$

hence

$$\frac{\partial\lambda}{\partial t}=\lambda\boldsymbol{\tau}\cdot\frac{\partial\mathbf{U}}{\partial s}.$$

If we introduce this result in the line integral, we obtain:

$$\int_{\mathrm{A_0B_0}} G\frac{\partial\lambda}{\partial t}\,\mathrm{d}p_0=\int_{\mathrm{A_0B_0}} G\boldsymbol{\tau}\cdot\frac{\partial\mathbf{U}}{\partial s}\lambda\,\mathrm{d}p_0$$

$$=\int_{\mathrm{AB}} g\boldsymbol{\tau}\cdot\frac{\partial\mathbf{U}}{\partial s}\,\mathrm{d}s.$$

As a result, we have finally:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{AB}} g\,\mathrm{d}s=\int_{\mathrm{AB}}\left(\frac{\mathrm{d}g}{\mathrm{d}t}+g\boldsymbol{\tau}\cdot\frac{\partial\mathbf{U}}{\partial s}\right)\mathrm{d}s,$$

where

$$\frac{\mathrm{d}g}{\mathrm{d}t}=\frac{\partial g}{\partial t}+\frac{\partial g}{\partial s}\boldsymbol{\tau}\cdot\mathbf{U}.$$

The time derivative of a line integral can thus be written as

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{AB}} g\,\mathrm{d}s=\int_{\mathrm{AB}}\left(\frac{\partial g}{\partial t}+\frac{\partial g}{\partial s}\boldsymbol{\tau}\cdot\mathbf{U}+g\boldsymbol{\tau}\cdot\frac{\partial\mathbf{U}}{\partial s}\right)\mathrm{d}s.}\qquad\text{[A1]}$$

Two particular cases of the time derivative of a line integral can now be given:

*Case 1*

Let us consider *a point M attached to the line C(t)*. Consequently, we have:

$$p=p(p_0).$$

An analogous proof leads to the following relation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{AB}} g\,\mathrm{d}s=\int_{\mathrm{AB}}\left(\frac{\partial g}{\partial t}+\frac{\partial g}{\partial s}\boldsymbol{\tau}\cdot\mathbf{U}_C+g\boldsymbol{\tau}\cdot\frac{\partial\mathbf{U}_C}{\partial s}\right)\mathrm{d}s,}\qquad\text{[A2]}$$

where $\mathbf{U}_C$ is defined by

$$\mathbf{U}_C\hat{=}\left.\frac{\partial\mathbf{OM}}{\partial t}\right|_{p\text{ fixed}}.$$

*Case 2*

The line considered now is a *material line $\mathscr{C}(t)$*. The proof is the same as in the general case, the velocity $\mathbf{V}$ of a fluid particle has to be substituted for $\mathbf{U}$:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{AB}} g\,\mathrm{d}s=\int_{\mathrm{AB}}\left(\frac{\partial g}{\partial t}+\frac{\partial g}{\partial s}\boldsymbol{\tau}\cdot\mathbf{V}+g\boldsymbol{\tau}\cdot\frac{\partial\mathbf{V}}{\partial s}\right)\mathrm{d}s.}\qquad\text{[A3]}$$

# APPENDIX B

*Differential Operators on a Line or on a Surface*

*B1. The line (see figure B1)*

- Divergence of a vector $\mathbf{B}$,

$$\mathrm{div}_l\,\mathbf{B}=\boldsymbol{\tau}\cdot\frac{\partial\mathbf{B}}{\partial s}.$$

Divergence of a tensor $\mathbb{T}$,

$$\mathrm{div}_l\,\mathbb{T}=\boldsymbol{\tau}\cdot\frac{\partial\mathbb{T}}{\partial s}.$$

- Gradient of a scalar field $f$,

$$\mathbf{grad}_l f=\frac{\partial f}{\partial s}\boldsymbol{\tau}.$$

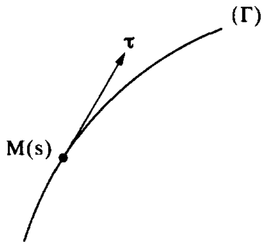


Figure B1. A line $(\Gamma)$.

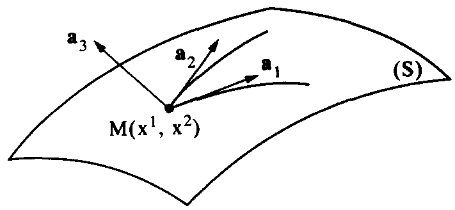


Figure B2. A surface $(S)$.

Gradient of a vector **B**,

$$\operatorname{grad}_l \mathbf{B} = \frac{\partial \mathbf{B}}{\partial s}\boldsymbol{\tau}.$$

*B2. The surface (see figure B2)*

- Divergence of a vector **B**,

$$\mathbf{B} = B^\alpha \mathbf{a}_\alpha + B^3 \mathbf{a}_3$$

and

$$\operatorname{div}_s \mathbf{B} = B^\alpha_{|\alpha} - b^\alpha_\alpha B^3,$$

where the subscript $|\alpha$ stands for the covariant derivative with respect to $x^\alpha$ and $b^\alpha_\alpha \triangleq 1/R_1 + 1/R_2$, with $R_1$ and $R_2$ as the principal radii of curvature of the surface $S$. Divergence of a tensor $\mathbb{T}$,

$$\mathbb{T} = T^{\alpha\beta}\mathbf{a}_\alpha \mathbf{a}_\beta + T^{\alpha 3}\mathbf{a}_\alpha \mathbf{a}_3 + T^{3\alpha}\mathbf{a}_3 \mathbf{a}_\alpha + T^{33}\mathbf{a}_3 \mathbf{a}_3$$

and

$$\operatorname{div}_s \mathbb{T} = (T^{\alpha\beta}_{|\beta} - b^\alpha_\lambda T^{3\lambda} - b^\nu_\nu T^{\alpha 3})\mathbf{a}_\alpha + (T^{3\beta}_{|\beta} + b_{\gamma\nu} T^{\gamma\nu} - b^\nu_\nu T^{33})\mathbf{a}_3.$$

- Gradient of a scalar field $f$,

$$\mathbf{grad}_s f = a^{\alpha\beta} f_{,\beta}\mathbf{a}_\alpha.$$

Gradient of a vector **B**,

$$\operatorname{grad}_s \mathbf{B} = [(B^\alpha_{|\beta} - b^\alpha_\beta B^3)\mathbf{a}_\alpha + (B^3_{|\beta} + b_{\gamma\beta} B^\gamma)\mathbf{a}_3]\mathbf{a}^\beta.$$

# APPENDIX C

*The Leibniz and Gauss Theorems for Surfaces and Segments*

The derivation of equations averaged over an area or a segment requires appropriate limiting forms of the Leibniz and Gauss theorems. Starting in each case from three-dimensional relations, formulas are obtained involving only quantities connected to the cross section or to the segment. We will suppose that the different quantities introduced are smooth enough.

*C1. The Leibniz and Gauss theorems for surfaces*

Let us consider a region surrounding a curve $(\Gamma)$ such that, at any point M on the curve, we associate a circular section $(S)$ of center M and radius $R(s)$. We consider the volume $V$ generated by the surface $(S)$ when M describes the curve $(\Gamma)$ between $\mathrm{M}_1$ and $\mathrm{M}_2$ (figure C1). The volume $V$ is thus limited by two cross sections $S_1$ and $S_2$ and the lateral surface $(\Sigma)$.

*C1.1. Gauss theorem.* The Gauss theorem applied to volume $V$ leads to

$$\iiint_V \operatorname{div} \mathbf{B}\, dv = \iint_\Sigma \mathbf{B}\cdot\mathbf{n}_\Sigma\, da + \iint_{S_1} \mathbf{B}\cdot\mathbf{n}_1\, da + \iint_{S_2} \mathbf{B}\cdot\mathbf{n}_2\, da, \qquad \text{[C1]}$$

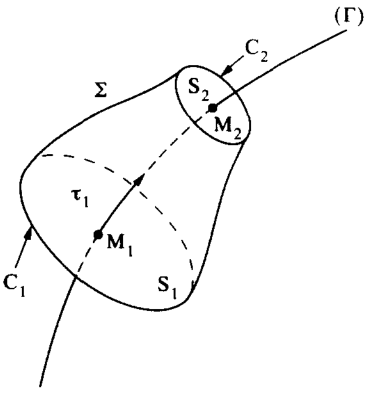


Figure C1. A thin filament.

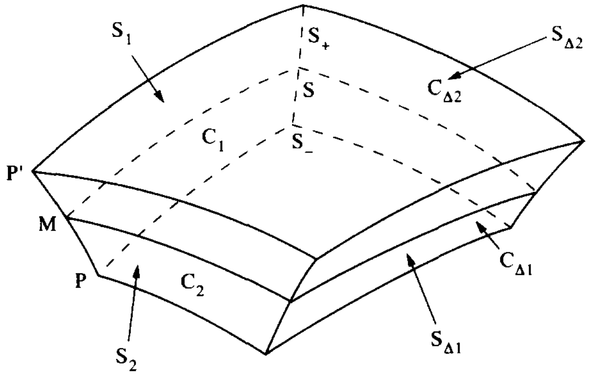


Figure C2. A thin membrane.

where $\mathbf{n}_\Sigma$ is the unit vector normal to $(S)$ directed away from the volume $V$ and $\mathbf{n}_\alpha$ $(\alpha = 1, 2)$ is the unit vector normal to $(S_\alpha)$ directed away from the volume $V$. Each term on the RHS of [C1] is computed successively by

$$\iiint_V \operatorname{div} \mathbf{B} \, \mathrm{d}v = \int_{M_1M_2} \left\{ \iint_{S(M,t)} \operatorname{div} \mathbf{B}\lambda \, \mathrm{d}a \right\} \mathrm{d}s, \tag{C2}$$

where $\lambda \triangleq 1 - x/\mathscr{R}$, and

$$\int_\Sigma \mathbf{B} \cdot \mathbf{n}_\Sigma \, \mathrm{d}a = \int_{M_1M_2} \left\{ \int_{C(M,t)} \mathbf{B} \cdot \left[ \left( 1 - \frac{R}{\mathscr{R}} \cos\varphi \right) \mathbf{l}_r - \frac{\partial R}{\partial s} \boldsymbol{\tau} \right] \mathrm{d}l \right\} \mathrm{d}s.$$

If $\boldsymbol{n}_C$ denotes the unit vector normal to the curve $(C)$ located in the cross-section plane:

$$\frac{\mathbf{B} \cdot \mathbf{n}_\Sigma}{\mathbf{n}_\Sigma \cdot \mathbf{n}_C} = \frac{B_r \left( 1 - \dfrac{R}{\mathscr{R}} \cos\varphi \right) - B_\tau \dfrac{\partial R}{\partial s}}{1 - \dfrac{R\cos\varphi}{\mathscr{R}}},$$

where

$$\mathbf{B} = B_r \mathbf{l}_r + B_\varphi \mathbf{l}_\varphi + B_\tau \boldsymbol{\tau}.$$

Consequently,

$$\iint_\Sigma \mathbf{B} \cdot \mathbf{n}_\Sigma \, \mathrm{d}a = \int_{M_1M_2} \left\{ \int_{C(M,t)} \frac{\mathbf{B} \cdot \mathbf{n}_\Sigma}{\mathbf{n}_\Sigma \cdot \mathbf{n}_C} \lambda \, \mathrm{d}l \right\} \mathrm{d}s \tag{C3}$$

and

$$\iint_{S_1} \mathbf{B} \cdot \mathbf{n}_1 \, \mathrm{d}a + \iint_{S_2} \mathbf{B} \cdot \mathbf{n}_2 \, \mathrm{d}a = -\iint_{S_1} \mathbf{B} \cdot \boldsymbol{\tau}_1 \, \mathrm{d}a + \iint_{S_2} \mathbf{B} \cdot \boldsymbol{\tau}_2 \, \mathrm{d}a.$$

Thus, one obtains:

$$\sum_{\alpha=1,2} \iint_{S_\alpha} \mathbf{B} \cdot \mathbf{n}_\alpha \, \mathrm{d}a = \int_{M_1M_2} \frac{\partial}{\partial s} \left( \iint_{S(M,t)} \mathbf{B} \cdot \boldsymbol{\tau} \, \mathrm{d}a \right) \mathrm{d}s. \tag{C4}$$

The combination of the different terms leads to a relation which holds for each part $M_1M_2$ of the line $(\Gamma)$. An original particular form of the Gauss theorem for the surfaces is then deduced:

$$\boxed{\iint_{S(M,t)} \operatorname{div} \mathbf{B}\lambda \, \mathrm{d}a = \frac{\partial}{\partial s} \left( \iint_{S(M,t)} \mathbf{B} \cdot \boldsymbol{\tau} \, \mathrm{d}a \right) + \int_{C(M,t)} \frac{\mathbf{B} \cdot \mathbf{n}_\Sigma}{\mathbf{n}_\Sigma \cdot \mathbf{n}_C} \lambda \, \mathrm{d}l,} \tag{C5}$$

where $\mathbf{n}_\Sigma$ denotes the unit vector normal to the surface $(\Sigma)$, $\mathbf{n}_C$ is the unit vector normal to the boundary $C(\mathrm{M}, t)$ of $S(\mathrm{M}, t)$ located in the cross-sectional plane; $\mathbf{n}_\Sigma$ and $\mathbf{n}_C$ are directed away from the volume $V$.

*C1.2. Leibniz theorem.* The Leibniz theorem applied to a volume $\mathcal{V}(\mathrm{M}_1, \mathrm{M}_2, t)$ limited by two cross sections $(S_1)$ and $(S_2)$ associated to two fixed values $p_1$ and $p_2$ of a parameter $p$, leads to

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} f\,\mathrm{d}v = \iiint_{\mathcal{V}} \frac{\partial}{\partial t} f\,\mathrm{d}v + \iint_{\Sigma} f\mathbf{v}_\Sigma \cdot \mathbf{n}_\Sigma\,\mathrm{d}a + \sum_{\alpha=1,2}\iint_{S(\mathrm{M},t)} f\mathbf{v}_\alpha \cdot \mathbf{n}_\alpha\,\mathrm{d}a; \qquad \text{[C6]}$$

$\mathbf{v}_\Sigma \cdot \mathbf{n}_\Sigma$ is the speed of displacement of the lateral surface $(\Sigma)$ and $\mathbf{v}_\alpha \cdot \mathbf{n}_\alpha$ is the speed of displacement of the surface $(S_\alpha)$ $(\alpha = 1, 2)$. With the method used above, one obtains an original particular form of the Leibniz theorem for a surface:

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\iint_{S(\mathrm{M},t)} f\lambda\,\mathrm{d}a + \left(\frac{\partial}{\partial s}\mathbf{U}_\Gamma\right)\cdot\boldsymbol{\tau}\iint_{S(\mathrm{M},t)} f\lambda\,\mathrm{d}a + (\mathbf{U}_\Gamma\cdot\boldsymbol{\tau})\frac{\partial}{\partial s}\iint_{S(\mathrm{M},t)} f\lambda\,\mathrm{d}a\\ &= \iint_{S(\mathrm{M},t)} \frac{\partial f}{\partial t}\lambda\,\mathrm{d}a + \frac{\partial}{\partial s}\iint_{S(\mathrm{M},t)} f(\mathbf{U}_s\cdot\boldsymbol{\tau})\,\mathrm{d}a + \int_{C(\mathrm{M},t)} f\frac{\mathbf{U}_\Sigma\cdot\mathbf{n}_\Sigma}{\mathbf{n}_\Sigma\cdot\mathbf{n}_C}\lambda\,\mathrm{d}l,\end{aligned}} \qquad \text{[C7]}$$

where $\mathbf{U}_\Gamma$ is the velocity of a point attached to the line $(\Gamma)$, $\mathbf{U}_s \cdot \boldsymbol{\tau}$ is the speed of displacement of the section $(S)$ and $\mathbf{U}_\Sigma \cdot \mathbf{n}_\Sigma$ is the speed of displacement of the lateral surface $(\Sigma)$.

*C2. The Leibniz and Gauss theorems for segments*

The segment considered, PP′ is normal to the surface $(S)$ at M (figure C2). The associated volume is limited by the surface $S_1$, $S_2$, $S_{\Delta 1}$ and $S_{\Delta 2}$ generated by the normals to $(S)$ intersecting the coordinates curves $C_1$, $C_2$, $C_{\Delta 1}$ and $C_{\Delta 2}$ corresponding, respectively, to $x^1$, $x^2$, $x^1 + \Delta x^1$ and $x^2 + \Delta x^2$ and by the surfaces $(S_+)$ and $(S_-)$ defined by $x_3 = h/2$ and $x_3 = -h/2$.

*C2.1. Gauss theorem.* The Gauss theorem applied to volume $V$ leads to

$$\iiint_V \operatorname{div}\mathbf{B}\,\mathrm{d}v = \sum_{\alpha=1,2}\iint_{S_\alpha}\mathbf{B}\cdot\mathbf{n}_\alpha\,\mathrm{d}a + \sum_{\alpha=1,2}\iint_{S_{\Delta\alpha}}\mathbf{B}\cdot\mathbf{n}_{\Delta\alpha}\,\mathrm{d}a + \iint_{S_+}\mathbf{B}\cdot\mathbf{n}_+\,\mathrm{d}a + \iint_{S_-}\mathbf{B}\cdot\mathbf{n}_-\,\mathrm{d}a. \qquad \text{[C8]}$$

The different vectors introduced are unit vectors normal to the corresponding surfaces $(S_i)$ and directed away from volume $V$.

As developed in section C1.1, each term on the RHS of [C8] is written as

$$\iiint_V \operatorname{div}\mathbf{B}\,\mathrm{d}v = \iint_S\left\{\int_{-h/2}^{+h/2}\operatorname{div}\mathbf{B}\mu\,\mathrm{d}x^3\right\}\mathrm{d}a. \qquad \text{[C9]}$$

By means of [D6] in appendix D:

$$\iint_{S_+}\mathbf{B}\cdot\mathbf{n}_+\,\mathrm{d}a + \iint_{S_-}\mathbf{B}\cdot\mathbf{n}_-\,\mathrm{d}a = \iint_{S_+}\mathbf{B}\cdot\mathbf{A}_3^+\,\mathrm{d}a - \iint_{S_-}\mathbf{B}\cdot\mathbf{A}_3^-\,\mathrm{d}a.$$

Consequently, one obtains:

$$\iint_{S_+}\mathbf{B}\cdot\mathbf{n}_+\,\mathrm{d}a + \iint_{S_-}\mathbf{B}\cdot\mathbf{n}_-\,\mathrm{d}a = \iint_S[\mathbf{B}\cdot(\mathbf{A}_1\wedge\mathbf{A}_2)]_{-h/2}^{+h/2}\frac{\mathrm{d}a}{\sqrt{a}}.$$

If $\mathbf{N}_\mathrm{P}$ denotes the unit vector normal to $(S_-)$ at P and $\mathbf{N}_{\mathrm{P}'}$ is the unit vector normal to $(S_+)$ at P′ outwardly directed:

$$[\mathbf{B}\cdot(\mathbf{A}_1\wedge\mathbf{A}_2)]_{x_3=-h/2} = \left(\frac{\mathbf{B}\cdot\mathbf{N}_\mathrm{P}}{\mathbf{N}_\mathrm{P}\cdot\mathbf{a}_3}\mu\sqrt{a}\right)_\mathrm{P} \quad \text{denoted by} \quad \frac{\mathbf{B}_\mathrm{P}\cdot\mathbf{N}_\mathrm{P}}{\mathbf{N}_\mathrm{P}\cdot\mathbf{a}_3}\mu_\mathrm{P}\sqrt{a},$$

then

$$\iint_{S_+} \mathbf{B}\cdot\mathbf{n}_+\,da + \iint_{S_-} \mathbf{B}\cdot\mathbf{n}_-\,da = \iint_S \left[\frac{\mathbf{B}_{P'}\cdot\mathbf{N}_{P'}}{\mathbf{N}_{P'}\cdot\mathbf{a}_3}\mu_{P'} - \frac{\mathbf{B}_P\cdot\mathbf{N}_P}{\mathbf{N}_P\cdot\mathbf{a}_3}\mu_P\right] da. \quad [C10]$$

According to Naghdi (1963), one has

$$\iint_{S_\alpha} \mathbf{B}\cdot\mathbf{n}_\alpha\,da = \int_{C_\alpha} v_\alpha \left\{\int_{-h/2}^{+h/2} \mu B^\alpha\,dx^3\right\} ds,$$

where $\mathbf{v}_\alpha$ is the unit vector normal to $S_\alpha$ at any point pertaining to $C$ and outwardly directed. As a result, one obtains:

$$\sum_{\alpha=1,2} \iint_{S_\alpha} \mathbf{B}\cdot\mathbf{n}_\alpha\,da + \sum_{\alpha=1,2} \iint_{S_{\Delta\alpha}} \mathbf{B}\cdot\mathbf{n}_{\Delta\alpha}\,da = \iint_S \mathrm{div}_s\left(\int_{-h/2}^{+h/2} \mu\mathbf{B}_\parallel\,dx^3\right) da,$$

where

$$\mathbf{B}_\parallel \triangleq (\mathbf{B}\cdot\mathbf{g}^\alpha)\mathbf{a}_\alpha \quad [C11]$$

and

$$\mathrm{div}_s\,\mathbf{B} = B^\alpha_{|\alpha} - 2HB^3 \quad \text{with} \quad \mathbf{B} = \sum_{\alpha=1}^{2} B^\alpha\mathbf{a}_\alpha + B^3\mathbf{a}_3,$$

where $\mathbf{g}^\alpha$ is defined as in appendix D and the subscript | stands for the covariant differentiation with respect to $a_{\alpha\beta}$ and $H$ is the mean curvature of the surface ($S$).

The combination of the different terms leads to a relation which holds for each part ($S$) of the surface. A particular form of the Gauss theorem for segments is then deduced:

$$\boxed{\int_{-h/2}^{+h/2} \mathrm{div}\,\mathbf{B}\mu\,dx^3 = \mathrm{div}_s \int_{-h/2}^{+h/2} \mu\mathbf{B}_\parallel\,dx^3 + \frac{\mu_{P'}\mathbf{B}\cdot\mathbf{N}_{P'}}{\mathbf{N}_{P'}\cdot\mathbf{a}_3} - \frac{\mu_P\mathbf{B}\cdot\mathbf{N}_P}{\mathbf{N}_P\cdot\mathbf{a}_3}.} \quad [C12]$$

In the following, the last two terms will be written as

$$\left[\frac{\mu\mathbf{B}\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2}.$$

*C2.2. Leibniz theorem.* The Leibniz theorem applied to the volume $\mathscr{V}(x^1, x^2, \Delta x^1, \Delta x^2, t)$ with fixed $x^1$, $x^2$, $\Delta x^1$ and $\Delta x^2$, leads to

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} f\,dv = \iiint_{\mathscr{V}} \frac{\partial f}{\partial t}\,dv + \iint_{\mathscr{A}} f\mathbf{V}_a\cdot\mathbf{n}_a\,da, \quad [C13]$$

where $\mathbf{V}_a\cdot\mathbf{n}_a$ is the speed of displacement of the surface $\mathscr{A}$, the boundary of $\mathscr{V}$.

With the same method as in section C2.1 an original particular form of the Leibniz theorem for the segments is derived:

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\int_{-h/2}^{+h/2} f\mu\,dx^3 + \mathrm{div}_s\,\mathbf{U}_s \int_{-h/2}^{+h/2} \mu f\,dx^3 + \mathbf{U}_s\cdot\mathbf{grad}_s\left(\int_{-h/2}^{+h/2}\mu f\,dx^3\right)\\ &= \int_{-h/2}^{+h/2} \mu\frac{\partial f}{\partial t}\,dx^3 + \mathrm{div}_s\left(\int_{-h/2}^{+h/2} \mu f\mathbf{U}_{\Sigma\parallel}\,dx^3\right) + \left[\mu f\frac{\mathbf{U}_{S_\pm}\cdot\mathbf{N}_\pm}{\mathbf{N}_\pm\cdot\mathbf{a}_3}\right]_{-h/2}^{+h/2},\end{aligned}} \quad [C14]$$

where $\mathbf{U}_s$ is the speed of the points on surface ($S$), $\mathbf{U}_{\Sigma\parallel} \triangleq (\mathbf{U}_\Sigma\cdot\mathbf{g}^\alpha)\mathbf{a}_\alpha$, $\mathbf{U}_{S_\pm}\cdot\mathbf{N}_\pm$ is the speed of displacement of the interface ($S_\pm$), $\mathbf{U}_{S_\pm}$ is the speed of a point located on the interface ($S_\pm$), $\mathbf{N}_\pm$ is the unit vector normal to the interface outwardly directed and

$$\mathrm{div}_s\,\mathbf{V} = V^\alpha_{|\alpha} - 2HV^3 \quad \text{with} \quad \mathbf{V} = \sum_{\alpha=1}^{2} V^\alpha\mathbf{a}_\alpha + V^3\mathbf{a}_3.$$

*C3. Generalization*

The limiting forms of Gauss's theorem can be generalized for tensors. Only the results will be given here.

For a surface:

$$\boxed{\iint_{S(s)} \operatorname{div} \mathbb{T}\lambda \, da = \frac{\partial}{\partial s} \iint_{S(s)} \mathbb{T} \cdot \mathbf{g}^1 \lambda \, da + \int_{C(s)} \frac{\mathbb{T} \cdot \mathbf{n}_\Sigma}{\mathbf{n}_\Sigma \cdot \mathbf{n}_C} \lambda \, dl,} \qquad [C15]$$

where $\mathbf{g}^1$ is defined by [D2].

For a segment:

$$\boxed{\int_{-h/2}^{+h/2} \operatorname{div} \mathbb{T}\mu \, dx^3 = \operatorname{div}_s\left(\int_{-h/2}^{+h/2} \mu(\mathbb{T} \cdot \mathbf{g}^\beta) \, dx^3 \mathbf{a}_\beta\right) + \left[\mu \frac{\mathbb{T} \cdot \mathbf{N}_\pm}{\mathbf{N}_\pm \cdot \mathbf{a}_3}\right]_{-h/2}^{+h/2}.} \qquad [C16]$$

Similarly, the limiting forms of the Leibniz theorem can be established for a vector:

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t} \iint_{S(s,t)} \mathbf{B}\lambda \, da + \left(\frac{\partial \mathbf{U}_\Gamma}{\partial s} \cdot \boldsymbol{\tau}\right) \iint_{S(s,t)} \mathbf{B}\lambda \, da + (\mathbf{U}_\Gamma \cdot \boldsymbol{\tau}) \frac{\partial}{\partial s} \iint_{S(s,t)} \mathbf{B}\lambda \, da \\ &\quad = \iint_{S(s,t)} \frac{\partial \mathbf{B}}{\partial t} \lambda \, da + \iint_{S(s,t)} \mathbf{B}(\mathbf{U}_s \cdot \boldsymbol{\tau}) \, da + \int_{C(s,t)} \mathbf{B} \frac{\mathbf{U}_\Sigma \cdot \mathbf{n}_\Sigma}{\mathbf{n}_\Sigma \cdot \mathbf{n}_C} \lambda \, dl \end{aligned}} \qquad [C17]$$

and

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t} \int_{-h/2}^{+h/2} \mathbf{B}\mu \, dx^3 + \operatorname{div}_s \mathbf{U}_s \int_{-h/2}^{+h/2} \mathbf{B}\mu \, dx^3 + \operatorname{grad}_s\left(\int_{-h/2}^{+h/2} \mathbf{B}\mu \, dx^3\right) \cdot \mathbf{U}_s \\ &= \int_{-h/2}^{+h/2} \frac{\partial \mathbf{B}}{\partial t} \mu \, dx^3 + \operatorname{div}_s \int_{-h/2}^{+h/2} \mathbf{B}\mathbf{U}_{\Sigma\parallel} \mu \, dx^3 + \left[\mu \mathbf{B} \frac{\mathbf{U}_{S\pm} \cdot \mathbf{n}_\pm}{\mathbf{n}_\pm \cdot \mathbf{a}_3}\right]_{-h/2}^{+h/2}. \end{aligned}} \qquad [C18]$$

# APPENDIX D

*Curves and Surfaces*

*D1. Space curves*

Let $(\Gamma)$ be a space curve (figure D1). The coordinates of any point M on this curve are functions of a single parameter. The arc length $s$ is taken as the parameter along the curve. (M; $\boldsymbol{\tau}$, $\mathbf{n}$, $\mathbf{b}$) denotes the Frenet–Serret frame. The Frenet formula can be written as

$$\frac{d\boldsymbol{\tau}}{ds} = \frac{\mathbf{n}}{\mathscr{R}}, \quad \frac{d\mathbf{n}}{ds} = -\frac{\boldsymbol{\tau}}{\mathscr{R}} - \frac{\mathbf{b}}{\mathscr{T}}, \quad \frac{d\mathbf{b}}{ds} = \frac{\mathbf{n}}{\mathscr{T}},$$

where $1/\mathscr{R}$ is the curvature of the curve and $1/\mathscr{T}$ is the torsion.

Let us consider a region surrounding the curve $(\Gamma)$ such that, at any point M on the curve, we associate a circular section $(S)$ of center M and radius $R(s)$ smaller than the radius of curvature $\mathscr{R}$ of $(\Gamma)$. The volume generated by the surface $(S)$ is now considered when M describes the curve $(\Gamma)$. Let $(\Sigma)$ be the boundary surface (figure D2). The cross section $(S)$ is referred to Cartesian coordinates or polar coordinates with respect to (M; $\mathbf{n}$, $\mathbf{b}$):

$$\mathbf{MP} = x\mathbf{n} + y\mathbf{b} = r \cos\varphi \, \mathbf{n} + r \sin\varphi \, \mathbf{b}.$$

Let $(s, x, y)$ or $(s, r, \varphi)$ be the curvilinear coordinate systems for this domain (Aris 1962).

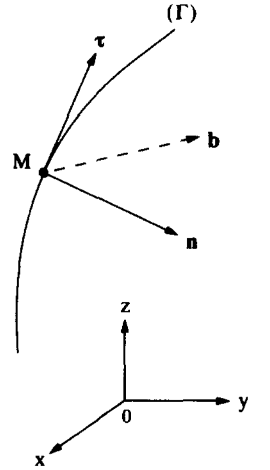

Figure D1. A space curve.

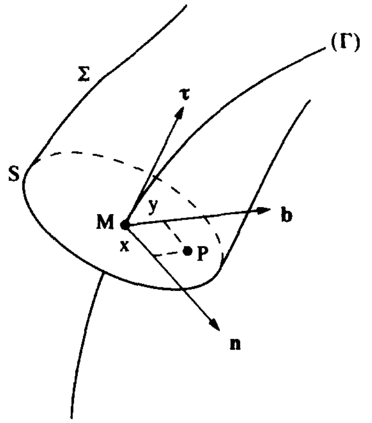

Figure D2. Volume generated by surface $(S)$.

The local base vectors associated to $(s, x, y)$ are defined by

$$\left.\begin{aligned} \mathbf{g}_1 &= \lambda \boldsymbol{\tau} + \frac{y}{\mathscr{T}}\mathbf{n} - \frac{x}{\mathscr{T}}\mathbf{b}, \\ \mathbf{g}_1 &= \mathbf{n}, \\ \mathbf{g}_3 &= \mathbf{b}, \\ \lambda &= 1 - \frac{x}{\mathscr{R}}. \end{aligned}\right\} \qquad [\mathrm{D1}]$$

The reciprocal basis denoted by $(\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)$ is defined by

$$\left.\begin{aligned} \mathbf{g}^1 &= \frac{1}{\lambda}\boldsymbol{\tau}, \\ \mathbf{g}^2 &= -\frac{y}{\mathscr{T}\lambda}\boldsymbol{\tau} + \mathbf{n}, \\ \mathbf{g}^3 &= +\frac{y}{\mathscr{T}\lambda}\boldsymbol{\tau} + \mathbf{b}. \end{aligned}\right\} \qquad [\mathrm{D2}]$$

Thus, an element of volume $\mathrm{d}V$ is given by

$$\mathrm{d}V = (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y = \sqrt{g}\, \mathrm{d}s\, \mathrm{d}x\, \mathrm{d}y,$$

where

$$\sqrt{g} = 1 - \frac{x}{\mathscr{R}}; \qquad [\mathrm{D3}]$$

$g$ being the determinant of the metric tensors $(g_{ij})$.

In what follows, all Latin indices take the values 1, 2, 3 and Greek indices the values 1, 2.

*D2. Surfaces in space*

Let $(S)$ be a space surface. A curvilinear coordinate system $(x^1, x^2)$ is defined for the surface. The local base vectors associated to $(S)$ at point M are $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$. We have the relations:

$$\mathbf{a}_\alpha \triangleq \frac{\partial \mathbf{M}}{\partial x^\alpha}; \quad \mathbf{a}_3 \triangleq \frac{\dfrac{\partial \mathbf{M}}{\partial x^1} \wedge \dfrac{\partial \mathbf{M}}{\partial x^2}}{\left|\dfrac{\partial \mathbf{M}}{\partial x^1} \wedge \dfrac{\partial \mathbf{M}}{\partial x^2}\right|};$$

$a_{\alpha\beta}$ is the metric tensor (Aris 1962) and $a$ is the determinant of the surface metric tensor $a_{\alpha\beta}$.

A region surrounding the surface $(S)$ is defined as follows. For every point M located on $(S)$, the normal to $(S)$ and two points P and P′ lying on this normal are associated such that:

$$\mathbf{MP'} = \mathbf{MP} = \frac{h}{2}\mathbf{a}_3,$$

where $h$ depends on $x^1$ and $x^2$ and $h/2$ is supposed to remain smaller than the smallest principal radius of curvature of the surface.

The domain $V$ in consideration, is the one generated by PP′ when M moves on the surface. The position vector of every point Q of this domain $V$ can be located by

$$\mathbf{OQ} = \mathbf{OM} + x\mathbf{a}_3, \quad |x^3| \leqslant \frac{h}{2};$$

$(x^1, x^2, x^3)$ is a curvilinear coordinate system for $V$. The local basis at Q $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ is defined by

$$\left.\begin{aligned} \mathbf{g}_1 &= \mathbf{a}_1 + x^3\frac{\partial \mathbf{a}_3}{\partial x^1}, \\ \mathbf{g}_2 &= \mathbf{a}_2 + x^3\frac{\partial \mathbf{a}_3}{\partial x^2}, \\ \mathbf{g}_3 &= \mathbf{a}_3. \end{aligned}\right\} \tag{D4}$$

The second fundamental form of the surface $(S)$ is given by (Naghdi 1963):

$$\left.\begin{aligned} \mathbf{g}_\alpha &= \mu_\alpha^\gamma \mathbf{a}_\gamma, \\ \mathbf{g}_3 &= \mathbf{a}_3, \end{aligned}\right\} \quad \text{where} \quad \mu_\alpha^\gamma = \delta_\alpha^\gamma - x^3 b_\alpha^\gamma. \tag{D5}$$

The reciprocal base vectors of the space coordinates are $(\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)$. An element of volume $\mathrm{d}V$ can be written as

$$\begin{aligned} \mathrm{d}V &= (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3 = \sqrt{g}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3, \\ \mathrm{d}V &= \mu\sqrt{a}\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3, \end{aligned} \tag{D6}$$

where $g$ is the determinant of the metric tensor $g_{ij}$, $a$ is the determinant of the metric tensor $a_{\alpha\beta}$, $\mu$ is the determinant of elements $\mu_\alpha^\gamma$, $\mu = 1 - 2Hx^3 + K(x^3)^2$, $H$ is the mean curvature of the surface, $\mu = \frac{1}{2}b_\alpha^\alpha$, $K$ is the Gaussian curvature, $K = b_2^2 b_1^1 - b_1^2 b_2^1$ and $\sqrt{a}\,\mathrm{d}x^1\,\mathrm{d}x^2$ is an element of area on the surface $(S)$.